

A note on high-energy scattering of open superstrings¹

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Abstract

We study the Regge and hard scattering limit of the one-loop amplitude for massless open string states in the type I theory. For hard scattering we find the exact coefficient multiplying the known exponential falloff in terms of the scattering angle, without relying on a saddle point approximation for the integration over the cross ratio. This bypasses the issues of estimating the contributions from flat directions, as well as those that arise from fluctuations of the gaussian integration about a saddle point. This result allows for a straightforward computation of the small-angle behavior of the hard scattering regime and we find complete agreement with the Regge limit at high momentum transfer, as expected.

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1 Introduction

The hard scattering limit of open strings at one-loop was studied long ago by Alessandrini, Amati and Morel [1] in the early days of string theory and subsequently by [2, 3]. They analyzed the (2+2) non-planar amplitude (i.e. the annulus with two external particles attached to each boundary) and found the characteristic exponential falloff now known for stringy amplitudes. The coefficient multiplying this exponential behavior, which contains dependence on the scattering angle, involves the well-known problem of inversion in the theory of elliptic modular functions. As a consequence of this, the angular dependence in the aforementioned coefficient could only be expressed in terms of an infinite series.

For the case of the planar and non-orientable amplitudes, Gross and Mañes [3] studied their behavior also in the hard scattering limit and found that, contrary to the (2+2) non-planar case, they do not possess a dominant saddle point in the interior of the integration region. Moreover, they were able to show that the dominant contributions come from the boundaries of this region, the one where the annulus shrinks to a point being the dominant in this case.

The study of the fixed-angle limit of the one-loop amplitude in different situations has been carried out by many authors [3–6], but as far as we aware of, we believe that the exact dependence on the scattering angle for the amplitude we study here has not been worked out in the literature in closed form.

We organize this short article as follows: in section 2 we review the calculation of the Regge limit of the sum of the planar and non-orientable diagrams of the type I theory. For this we also compute its large momentum transfer limit ($|t| \rightarrow \infty$) in order to make a comparison with the small angle behavior of the hard scattering limit which we also review in this section. In section 3, by making use of an identity found in [9], we compute the exact form of the coefficient that multiplies the exponential falloff of the hard scattering amplitude. This permits a straightforward evaluation of the the hard scattering amplitude in the limit where $t \ll s$, which indeed matches with the Regge behavior computed in section 2.

2 High-energy scattering of the type I open superstrings

2.1 Regge behavior at one-loop

We begin by computing the Regge limit, i.e. we take $s \rightarrow -\infty$ with t held fixed of the one-loop amplitude for type I open superstrings. The details of the calculation are basically the same as the ones for the NS+ string computed in [7] with the only difference being the nature of the cancellations of divergences due to the propagation of closed string tachyons and dilatons. In the NS+ model, the remnants of closed string tachyon divergences were cancelled by the inclusion of a counterterm which, after analytic continuation using the GNS regulator, turned out to be zero in the Regge limit. The “would-be” subleading divergences due to closed string dilatons were simply absent with the inclusion of Dp -

branes as long as $p < 8$.

The amplitude for four massless vector states is much simpler in the superstring compared to the NS+ model, because in the former the full polarization structure can be factored out of the loop integration, whereas in the latter each combination of polarization vectors must be worked out separately. For the SO(32) gauge group the planar and non-orientable diagrams combine to give a finite amplitude and we focus our attention on this case here. The amplitude for each diagram (planar and non-orientable) was computed long time ago (see for instance [8]) and for the SO(32) gauge group they can be combined as

$$A_P + A_N = 16\pi^3 g^4 G_P K \int_0^1 \frac{dq}{q} [F(q^2) - F(-q^2)] \quad (1)$$

with

$$\begin{aligned} F(q^2) &= \int_R \prod_{i=1}^3 d\theta_i \prod_{i<j} \psi(\theta_{ji})^{2\alpha' k_i \cdot k_j} \\ \psi(\theta) &= \sin \theta \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos 2\theta + q^{4n}}{(1 - q^{2n})^2} \end{aligned} \quad (2)$$

and K is the kinematic factor which can be found, for example in [8]. The region of integration R is given by $0 < \theta_2 < \theta_3 < \theta_4 < \pi$, $\theta_{ji} \equiv \theta_j - \theta_i$, and G_P is the group theory factor $G_P = \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4)$. We can now go ahead and compute the behavior of this expression for $s \rightarrow -\infty$ holding t fixed. In this limit, the amplitude is dominated by the region $\theta_2 \sim \theta_3$ and $\theta_4 \sim \pi$. Writing

$$\prod_{i<j} \psi(\theta_{ji})^{2\alpha' k_i \cdot k_j} = \left[\frac{\psi(\theta_{43})\psi(\theta_2)}{\psi(\theta_{42})\psi(\theta_3)} \right]^{-\alpha' s} \left[\frac{\psi(\theta_{41})\psi(\theta_{32})}{\psi(\theta_{42})\psi(\theta_3)} \right]^{-\alpha' t} \quad (3)$$

this implies that we need the following approximations:

$$\begin{aligned} \left[\frac{\psi(\theta_{43})\psi(\theta_2)}{\psi(\theta_{42})\psi(\theta_3)} \right]^{-\alpha' s} &\sim \exp\{-\alpha' s \theta_{32} (\pi - \theta_4) (\ln \psi)''\} \\ \left[\frac{\psi(\theta_{41})\psi(\theta_{32})}{\psi(\theta_{42})\psi(\theta_3)} \right]^{-\alpha' t} &\sim \left(\frac{\theta_{32}(\pi - \theta_4)}{\psi^2(\theta_3)} \right)^{-\alpha' t} \end{aligned} \quad (4)$$

The dominant term in K for this limit is

$$K \sim \frac{1}{4} \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_4 s^2 \quad (5)$$

Using the approximations above, we see that we need to compute the integral

$$I \equiv \int_0^\epsilon dx \int_0^\epsilon dy (xy)^a e^{-xyk} \quad (6)$$

in the limit when $k \rightarrow \infty$. After some algebra this becomes

$$\begin{aligned} I &= k^{-a-1} \left[\ln(\epsilon^2) \int_0^{\epsilon^2 k} dz z^a e^{-z} + \ln k \int_0^{\epsilon^2 k} dz z^a e^{-z} - \int_0^{\epsilon^2 k} dz z^a e^{-z} \ln z \right] \\ &\sim k^{-a-1} [\Gamma(1+a) \ln k - a^{-1} \Gamma'(1+a)] + \mathcal{O}(k^{-a-2} \ln k) \end{aligned} \quad (7)$$

Thus, the Regge limit of the amplitude is

$$A_P + A_M \sim g^4(-\alpha's)^{1+\alpha't}\Gamma(-\alpha't)\ln(-\alpha's)\Sigma(t) \quad (8)$$

where

$$\Sigma(t) \equiv \alpha't \int_0^1 \frac{dq}{q} \int_0^\pi \left(\psi^{2\alpha't}[-\ln \psi'']^{\alpha't-1} - \psi_N^{2\alpha't}[-\ln \psi_N'']^{\alpha't-1} \right) \quad (9)$$

and $\psi_N(\theta, q^2) = \psi(\theta, -q^2)$. This completes the calculation of the asymptotic behavior of the amplitude in the Regge limit. Notice also that the function $\Sigma(t)$ gives the one-loop correction to the open string Regge trajectory. Since

$$(\beta(t) + \delta\beta)s^{\alpha(t)+\delta\alpha} \approx \beta s^{\alpha(t)} + \beta s^{\alpha(t)}\delta\alpha \log s + \delta\beta s^{\alpha(t)}, \quad (10)$$

the new trajectory is $\alpha(t)_{\text{new}} = 1 + \alpha't + g^2\Sigma(t)$. Given that we are also interested in seeing how both high-energy regimes (Regge and hard scattering) coincide, we need to extract the large t limit of $\Sigma(t)$. In order to do so re-write the integral as

$$\Sigma(t) \equiv \alpha't \int_0^1 \frac{dq}{q} \int_0^\pi \left(e^{\alpha't \ln(-\psi^2[\ln \psi]'')} [-\ln \psi'']^{-1} - e^{\alpha't \ln(-\psi_N^2[\ln \psi_N]'')} [-\ln \psi_N'']^{-1} \right) \quad (11)$$

thus the integral is dominated by the critical points of $\ln(-\psi^2[\ln \psi]'')$ and $\ln(-\psi_N^2[\ln \psi_N]'')$. Notice that now we only have a two-dimensional integration region, for which the critical points should be easier to analyze in principle. The leading contribution comes from the $q \sim 0$ region, thus we also need the approximations:

$$\begin{aligned} [-\ln \psi]'' &\sim [-\ln \psi_N]'' \sim \csc^2 \theta \\ \ln(-\psi^2[\ln \psi]'') &\sim -\ln(-\psi_N^2[\ln \psi_N]'') \sim 16q^2 \sin^4 \theta \end{aligned} \quad (12)$$

Notice that the regions $\theta \sim 0, \pi$ also produce important contributions to the integral for large t and need to be analyzed separately. For this purpose we would need the corresponding asymptotic expressions for the functions ψ and ψ_N and to integrate over the full range $0 < q < 1$. We will come back to this point at the end of this section and we will find that these regions produce subleading behavior with respect to the contribution coming from $q \sim 0$. For small q , $\Sigma(t)$ becomes

$$\Sigma(t) \sim i\alpha't \int_0^\epsilon \frac{dq}{q} \int_0^\pi d\theta \sin^2 \theta \left(e^{i16q^2 \sin^4 \theta \alpha't} - e^{-i16q^2 \sin^4 \theta \alpha't} \right) \quad (13)$$

Note that we have also defined the integral above by analytical continuation ($t \rightarrow it$) as in [3]. Therefore, we wish to obtain the large $|t|$ behavior of the expression

$$\Sigma(t) \sim i\alpha't \int_\delta^{\pi-\delta} d\theta \sin^2 \theta \int_0^\epsilon \frac{dq}{q} \left(e^{itaq^2} - e^{-itaq^2} \right) \quad (14)$$

for fixed ϵ with $a = 16 \sin^4 \theta$. We have also introduced the cutoff δ to stress that we need to examine the contributions from the regions where $\theta \sim 0, \pi$ separately. Performing the change $atq^2 \equiv u$ we have

$$\Sigma(t) \sim i\alpha't \int_\delta^{\pi-\delta} \sin^2 \theta d\theta i \int_0^{\epsilon^2 ta} \frac{du}{u} \sin u \quad (15)$$

Since ϵ is small but fixed we can take the upper limit of the u integral to be ∞ in the $|t| \rightarrow \infty$ limit. Also in this limit the θ dependence in the u integral disappears which allows us to send δ to zero, thus

$$\Sigma(t) \sim -\alpha't \int_0^\pi \sin^2 \theta d\theta \int_0^\infty \frac{du}{u} \sin u = -\alpha't \frac{\pi^2}{4} \quad (16)$$

Therefore, continuing back to $t \rightarrow -it$ we have

$$\Sigma(t) \sim i\alpha't \quad \text{as } t \rightarrow -\infty \quad (17)$$

Finally, as $t \rightarrow -\infty$, combining equations (8) and (17) yields

$$\begin{aligned} A_P + A_M &\sim i(-\alpha's)^{1+\alpha't} \Gamma(-\alpha't) \ln(-\alpha's) \alpha't \\ &= i(-\alpha's)^{1+\alpha't} \Gamma(1-\alpha't) \ln(-\alpha's) \end{aligned} \quad (18)$$

We could use Stirling's approximation $\Gamma(1-\alpha't) \sim \sqrt{2\pi}(-\alpha't)^{1/2-\alpha't} e^{\alpha't}$ valid for $-\alpha't \gg 1$, which yields

$$A_P + A_M \sim i(-\alpha's)^{1+\alpha't} (-\alpha't)^{1/2-\alpha't} e^{\alpha't} \ln(-\alpha's) \quad (19)$$

To conclude, we take a moment to analyze the regions where $\theta \sim 0, \pi$ which are also important as $|t|$ becomes large. Using the following expression for the logarithm of ψ

$$\ln \psi(\theta) = \ln \sin \theta + 2 \sum_{n=1}^{\infty} \frac{1}{m} \frac{q^{2m}}{1 - q^{2m}} (1 - \cos 2m\theta) \quad (20)$$

one can see that

$$\ln(-\psi^2[\ln \psi]') \sim -\ln(-\psi_N^2[\ln \psi_N]') \sim \mathcal{O}(\theta^4) \quad (21)$$

Thus, the main contribution at large t comes from the region where θ is of the order of $\sim (-\alpha't)^{-1/4}$. A rough estimation from these regions gives $\Sigma(t) \sim (-\alpha't)^{-3/4}$ which is subleading with respect to the $q \sim 0$ contribution given in (17).

2.2 Hard scattering at one loop

The high-energy limit at fixed scattering angle for the one-loop amplitude was first computed by [1] in the early days of string theory in the context of the old dual resonance models. There, the computation was done for the non-planar amplitude which had a dominant saddle point in the interior of the integration region. In [3], Gross and Mañes showed that only the (2+2) non-planar amplitude (i.e. the amplitude with two particles on each boundary of the annulus) has a saddle point in the interior of the region of integration. The planar, non-orientable and the (3+1) non-planar amplitudes do not possess a dominant saddle point in the interior, but points in the boundary of the region do give sub-dominant contributions (with respect to the (2+2) non-planar) from the boundaries of the region of integration. They also showed that the leading contribution for the sum of the planar and non-orientable diagrams comes from the region where $q \sim 0$ and the cross

ratio $x \equiv \frac{\sin \theta_2 \sin \theta_{43}}{\sin \theta_{42} \sin \theta_3}$ is approximately $(1 + t/s)^{-1}$ in equation (1). We begin this section by re-calculating the leading behavior known in the literature using the saddle point approximation for the cross-ratio although using a different set of integration variables [12] where $\theta_2 \rightarrow x$, $\theta_3 \rightarrow r \equiv \sin \theta_{43} / \sin \theta_3$. Starting from equations (1) and (2) the relevant factor in the integrand in this limit is

$$\prod_{i < j} \psi^{2\alpha' k_i \cdot k_j} = \exp\{-\alpha' s V_\lambda\} \quad (22)$$

where

$$V_\lambda \equiv \ln x - \lambda \ln(1 - x) + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1 - q^{2n}} (S_n - \lambda T_n) \quad (23)$$

$$x \equiv \frac{\sin \theta_2 \sin \theta_{32}}{\sin \theta_{42} \sin \theta_3} \quad (24)$$

$$\begin{aligned} S_n &\equiv 2 \cos n(\theta_2 - \theta_{43}) [\cos n(\theta_{42} + \theta_3) - \cos n(\theta_2 + \theta_{43})] \\ T_n &\equiv 2 \cos n(\theta_{42} + \theta_3) [\cos n(\theta_2 - \theta_{43}) - \cos n(\theta_2 + \theta_{43})] \end{aligned} \quad (25)$$

and $\lambda = -t/s$. Expanding the function V_λ about the critical region mentioned above yields

$$e^{-\alpha' s V_\lambda} \approx e^{-\mathcal{E}_0} e^{-\alpha' s [\frac{(1-\lambda)^3}{2\lambda} (x-x_c)^2 \pm 2q^2 (S_1 - \lambda T_1)]} \quad (26)$$

where

$$\begin{aligned} \mathcal{E}_0 &\equiv \alpha' |s| [\lambda \ln(-\lambda) + (1 - \lambda) \ln(1 - \lambda)] \\ &= \alpha' s \ln(-\alpha' s) + \alpha' t \ln(-\alpha' t) + \alpha' u \ln(\alpha' u) \end{aligned} \quad (27)$$

In the $|s| \rightarrow \infty$ limit, the integration over x can be approximated by a gaussian giving

$$\int_{-\infty}^{\infty} dx e^{-\alpha' s \frac{(1-\lambda)^3}{2\lambda} (x-x_c)^2} \sim \sqrt{\frac{-2\pi\lambda}{(1-\lambda)^3}} (-\alpha' s)^{-1/2} \quad (28)$$

The integral over q is dominated by the small q region which, after analytic continuation to $s \rightarrow is$ behaves as

$$\int_0^\epsilon \frac{dq}{q} \left(e^{2i\alpha' s q^2 (S_1 - \lambda T_1)} - e^{-2i\alpha' s q^2 (S_1 - \lambda T_1)} \right) \sim \frac{i\pi}{2} \quad (29)$$

result which we already encountered in (13). All in all, for the coefficient of $\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3$, we obtain:

$$\begin{aligned} A_P + A_M &\sim su e^{-\mathcal{E}_0} \sqrt{\frac{-2\pi\lambda}{(1-\lambda)^3}} (-\alpha' s)^{-1/2} F(\lambda) \\ &\sim s^2 (1 + t/s) e^{-\mathcal{E}_0} (-\alpha' t)^{1/2} (-\alpha' s)^{-1/2} (1 + t/s)^{-3/2} (-\alpha' s)^{-1/2} F(\lambda) \\ &\sim (-\alpha' s)^{3/2} e^{-\mathcal{E}_0} (-\lambda)^{1/2} (1 - \lambda)^{-1/2} F(\lambda) \end{aligned} \quad (30)$$

which shows the usual exponential suppression $e^{-\mathcal{E}_0}$ factor and where the function $F(\lambda)$ is given by

$$F(\lambda) = \int_0^\infty dr \int_0^\pi d\theta \frac{r \sin^2 \theta}{(r^2 + 2r \cos \theta + 1)(r^2(1 - \lambda)^2 + 2r(1 - \lambda) \cos \theta + 1)} \quad (31)$$

A few remarks are important to note about this integral. Since $-\infty < \lambda < 0$ it is convergent in this entire range but it diverges for $\lambda = 0$. As λ gets closer to zero, the integral becomes larger and larger and we need to estimate how it diverges in order to extract the correct small λ behavior. As we will show in the next section, we have that

$$F(\lambda) \sim -2 \ln(-\lambda) + 2 \ln(1 - \lambda) \sim 2 \ln(-\alpha' s) \quad \text{for } s \gg t \quad (32)$$

which provides the logarithm that appears in the Regge limit of the amplitude in (19). Writing the exponential factor as

$$e^{-\mathcal{E}_0} = (-\alpha' s)^{\alpha' t} (-\alpha' t)^{-\alpha' t} (1 + t/s)^{\alpha' s + \alpha' t} \quad (33)$$

we have

$$A_P + A_M \sim i(-\alpha' s)^{1+\alpha' t} (-\alpha' t)^{1/2-\alpha' t} (1 + t/s)^{\alpha' s + \alpha' t - 1/2} F(\lambda) \quad (34)$$

which completes the hard scattering limit of the one-loop amplitude.

3 Recovery of the Regge limit

The high-energy behavior at fixed angle given in Eq. (34) uses a gaussian approximation around the dominant saddle point given by $x_c = (1 - \lambda)^{-1}$. We will now calculate this limit using a different method which does not require the gaussian approximation but instead we will compute the integral over the x variable in an exact closed form. However, we still need to approximate the exponent for small q but this is not too serious since this is the only place in the q integration where there is dominant critical point [3]. One could regard the calculation we perform in this section as a computation of the gaussian approximation including all the possible fluctuations around the saddle. This allows us to bypass the issue of computing the contributions coming any other region in the θ_k integrations since we will be computing this triple integral in exact form. Starting from (1), we obtain

$$\begin{aligned} \prod_{i < j} \psi(\theta_{ji})^{2\alpha' k_i \cdot k_j} &= e^{-\alpha' s V_\lambda} \approx e^{-\alpha' s [\ln x - \lambda \ln(1-x) + 2q^2(S_1 - \lambda T_1)]} \\ &\approx x^{-\alpha' s} (1-x)^{-\alpha' t} e^{-2\alpha' s q^2(S_1 - \lambda T_1)} \end{aligned} \quad (35)$$

Notice that this time we are not expanding the function $\ln x - \lambda \ln(1-x)$ about the saddle point x_c . The small q contribution to the total amplitude can be written as

$$A_P + A_N \sim \alpha'^2 s u \int \prod_k d\theta_k x^{-\alpha' s} (1-x)^{-\alpha' t} \int_0^\epsilon \frac{dq}{q} \left[e^{-2\alpha' s q^2(S_1 - \lambda T_1)} - e^{2\alpha' s q^2(S_1 - \lambda T_1)} \right] \quad (36)$$

where we have included the overall $\alpha'^2 su$ coefficient coming from the $\epsilon_1 \epsilon_4 \epsilon_2 \epsilon_3$ structure. We have already encountered the expression for the q integral above with the very satisfying result that it does not depend on the coefficient of q^2 in the exponent, therefore it does not bring an angular dependence from the combination $S_1 - \lambda T_1$ which will allow us to perform an exact evaluation of the integration over the θ_k variables. The integral

$$I \equiv \int_0^\pi d\theta_4 \int_0^{\theta_4} d\theta_3 \int_0^{\theta_3} d\theta_2 x^{-\alpha' s} (1-x)^{-\alpha' t} \quad (37)$$

was evaluated long ago by Green and Schwarz [9] in the context of proving that dilaton tadpole divergences could be absorbed in a renormalization of the Regge slope α' . This was realized before it was recognized that this divergence is absent for the $SO(32)$ gauge group. We simply quote the answer here

$$I = \int \prod_k d\theta_k x^{-\alpha' s} (1-x)^{-\alpha' t} \propto \frac{1}{\alpha'} \frac{\partial}{\partial \alpha'} \left[\alpha' \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' t)}{\Gamma(1 - \alpha' s - \alpha' t)} \right] \quad (38)$$

Using this and the result for the integral over q given in eq. (29) we have

$$A_P + A_N \sim i \alpha'^2 su \frac{1}{\alpha'} \frac{\partial}{\partial \alpha'} \left[\alpha'^2 \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' t)}{\Gamma(1 - \alpha' s - \alpha' t)} \right] \quad (39)$$

We can now take the limit $s, t \rightarrow -\infty$ holding t/s fixed directly inside the brackets to obtain

$$\begin{aligned} A_P + A_N &\sim i \alpha'^2 su \frac{1}{\alpha'} \frac{\partial}{\partial \alpha'} \left[\alpha'^2 (-\alpha' s)^{-1+\alpha' t} (-\alpha' t)^{-1/2-\alpha' t} (1+t/s)^{-1/2+\alpha' s+\alpha' t} \right] \\ &\sim i (-\alpha' s)^{1/2} (-\lambda)^{-1/2-\alpha' t} (1-\lambda)^{1/2+\alpha' s+\alpha' t} [1 + 2\alpha' s (\lambda \ln(-\lambda) + (1-\lambda) \ln(1-\lambda))] \end{aligned} \quad (40)$$

Taking again $\alpha' s \gg 1$, we end up with

$$A_P + A_N \sim i (-\alpha' s)^{3/2} (-\lambda)^{-1/2} (1-\lambda)^{1/2} e^{\alpha' s [\lambda \ln(-\lambda) + (1-\lambda) \ln(1-\lambda)]} [\lambda \ln(-\lambda) + (1-\lambda) \ln(1-\lambda)]$$

To recover the Regge behavior we take $s \gg t$ above. The exponential becomes

$$e^{\alpha' s [\lambda \ln(-\lambda) + (1-\lambda) \ln(1-\lambda)]} = (-\lambda)^{-\alpha' t} (1-\lambda)^{\alpha' s+\alpha' t} \sim (-\alpha' s)^{\alpha' t} (-\alpha' t)^{-\alpha' t} e^{\alpha' t} \quad (41)$$

and the last factor becomes

$$\begin{aligned} [\lambda \ln(-\lambda) + (1-\lambda) \ln(1-\lambda)] &\sim \lambda \ln(-\lambda) = -t/s [\ln(-\alpha' t) - \ln(-\alpha' s)] \\ &\sim (-\alpha' t) (-\alpha' s)^{-1} \ln(-\alpha' s) \end{aligned} \quad (42)$$

Therefore, the Regge limit at high t is

$$\begin{aligned} A_P + A_N &\sim i (-\alpha' s)^{3/2} (-\lambda)^{-1/2} (-\alpha' s)^{\alpha' t} (-\alpha' t)^{-\alpha' t} e^{\alpha' t} (-\alpha' t) (-\alpha' s)^{-1} \ln(-\alpha' s) \\ &\sim i (-\alpha' s)^{1+\alpha' t} (-\alpha' t)^{1/2-\alpha' t} e^{\alpha' t} \ln(-\alpha' s) \end{aligned} \quad (43)$$

which is exactly the result we found in (19).

We finish this section by showing that the result in (43) can also be obtained from the

approximate expression in (34) by analyzing the small λ behavior of $F(\lambda)$ as anticipated in (32). We believe it is instructive to do this because we are also interested in the small λ behavior of the hard scattering limit of the NS+ string model in the context of [7, 13, 14] where we cannot afford the luxury of having an exact expression for the coefficient of the exponential falloff. For convenience we write this integral here again

$$F(\lambda) = \int_0^\infty dr \int_0^\pi d\theta \frac{r \sin^2 \theta}{(r^2 + 2r \cos \theta + 1)(r^2(1 - \lambda)^2 + 2r(1 - \lambda) \cos \theta + 1)} \quad (44)$$

As mentioned above, $F(\lambda)$ diverges as $\lambda \rightarrow 0$. The only singular region in this limit is $\theta \sim \pi$ and $r \sim 1$. It is straightforward to see this by recalling that, in terms of the cross ratio x , the dominant saddle point is given by $x_c = (1 - \lambda)^{-1}$. This perfectly matches with the fact that the Regge behavior of the amplitude is obtained from the region $\theta_2 \sim \theta_3$, $\theta_4 \sim \pi$ since $x \sim \theta_{32}(\pi - \theta_4)$ which gives the leading behavior [7, 10]. Thus, the Regge limit occurs when $x \rightarrow 1$. Therefore in the small scattering angle limit, the integral above is singular where $\theta \sim \pi$, $r \sim x$, thus

$$\begin{aligned} F(\lambda) &\sim \int_{x-\delta}^{x+\delta} dr \int_{\pi-\epsilon}^\pi d\theta \frac{(\pi - \theta)^2}{((x - 1)^2 + x(\pi - \theta)^2)((r/x - 1)^2 + (\pi - \theta)^2)} \\ &\sim 2 \int_0^\epsilon \frac{\theta}{(x - 1)^2 + x\theta^2} = -2 \ln |1 - x| + \ln((1 - x)^2 + \epsilon^2) \end{aligned} \quad (45)$$

Therefore, as $\lambda \rightarrow 0$ for fixed ϵ , we have

$$F(\lambda) \sim -2 (\ln(-\lambda) - \ln(1 - \lambda)) \sim 2 \ln(-\alpha' s) \quad (46)$$

as anticipated in (32).

4 Conclusions

By studying the hard scattering limit of the sum of the one-loop planar and non-orientable diagrams we found the exact dependence in the scattering angle that multiplies the known exponential suppression at high energies. This avoids the issue of having to estimate the contributions from flat directions in the angular integrals and the fluctuations around the saddle point, since we have at our disposal an exact result from the angular integral in a closed form. This allowed us to compare both, the hard scattering and Regge regimes of the amplitude, since they should coincide in the limit of high-momentum transfer of the latter regime. We indeed confirmed that this matching occurs by making use of the closed form of the angular integrals given in (38). We were also able to obtain this result from the approximate expression (34) by analyzing the behavior of the integral $F(\lambda)$ in (31) as $\lambda \rightarrow 0$.

An immediate extension of this work would be to allow the open strings to be attached to smaller dimensional Dp -brane (here we considered the case of a space-filling D-brane) where the small q behavior can be analyzed separately for the planar and non-orientable diagrams since the amplitudes are finite as long as $p < 8$. It would also be interesting to check if our results here could be also applied to the situation studied in [4] where they analyzed the case where the two colliding open strings lived on different D-branes separated by a fixed distance.

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